

# CENTRALIZERS OF TOEPLITZ OPERATORS WITH POLYNOMIAL SYMBOLS

AKAKI TIKARADZE

**ABSTRACT.** In this note we describe centralizers of Toeplitz operators with polynomial symbols on the Bergman space. As a consequence it is shown that if an element of the norm closed algebra generated by all Toeplitz operators commutes with a Toeplitz operator of a nonconstant polynomial, then this element is a Toeplitz operator of a bounded holomorphic function.

Following usual notation,  $L_a^2(D)$  will denote the Bergman space of the square integrable holomorphic functions on the open unit disk  $D = \{z \in \mathbb{C}, |z| < 1\}$ , and  $H^\infty(D)$  will denote the set of bounded holomorphic functions on  $D$ . Recall that  $L_a^2(D)$  is a Hilbert space under the inner product  $\langle f, g \rangle = \int_D f \bar{g} dA$  with respect to the standard Lebesgue measure with measure of the unit disc being 1. Elements  $\sqrt{n+1}z^n, n \geq 0$  form an orthonormal basis of  $L_a^2(D)$ . Recall also that for any bounded function  $f \in L^\infty(D)$ , one can define a bounded linear operator, called the Toeplitz operator  $T_f : L_a^2 \rightarrow L_a^2$  with symbol  $f$  defined as follows:  $T_f(g) = P(fg), g \in L_a^2(D)$ , where  $P$  is the orthogonal projection from  $L^2(D)$  to  $L_a^2(D)$ . We will consider a  $C^*$ -algebra generated by all Toeplitz operators, which becomes a  $C^*$ -subalgebra of the algebra of all bounded operators on  $L_a^2(D)$ . We will refer to this algebra as the Toeplitz algebra.

Cuckovic [C] and Cuckovic-Fan [CF] proved that centralizers of  $T_h$  in the Toeplitz algebra are Toeplitz operators with bounded analytic symbols, provided that  $h = z^m$  for some  $m > 0$  [C], or  $h = z + \sum_{i=2}^n a_i z^i, a_i \geq 0$  for all  $i$ . Motivated by these results, we show the following

**Theorem 0.1.** *Let  $S : L_a^2(D) \rightarrow L_a^2(D)$  be a bounded linear operator which commutes with  $T_{h(z^m)}$ , where  $h(z)$  is a polynomial which is not of the form  $h_1(z^l)$ , with  $h_1$  a polynomial and  $l$  a positive integer  $l > 1$ . Then,  $[S, T_{z^m}] = 0$ . In particular, if in addition  $S$  is compact, then  $S = 0$ .*

**Corollary 0.1.** *Let  $f(z)$  be an arbitrary nonconstant polynomial, if  $S$  is an element of the Toeplitz algebra which commutes with  $T_f$ , then  $A = T_g$ , for some  $g \in H^\infty(D)$ .*

The proof will follow closely ideas of Cuckovic [C], and Cuckovic-Fan [CF]. The key step is the following.

**Proposition 0.1.** *Let  $h(z) \in \mathbb{C}[z]$  be a polynomial which may not be written as a polynomial in  $z^m$  for any  $m > 1$ . Then there is a nonempty open set  $U \subset D$ , such that  $\frac{h(z)-h(w)}{z-w} \neq 0$ , for all  $w \in U, z \in \bar{D}$ .*

*Proof.* The open mapping property of  $h(z)$  implies that the boundary of  $h(\bar{D})$  is a subset of  $h(S^1)$  (where  $S^1$  is the unit circle, the boundary of  $D$ ) and is disjoint from  $h(D)$ . Since a theorem of Quine [Q] states that there are only finitely many pairs  $(z, w)$ , such that  $z \neq w, z, w \in S^1, h(z) = h(w)$ , we may conclude that there is a point  $w \in S^1$ , such that  $h(w)$  belongs to the boundary of  $h(\bar{D})$ ,  $\frac{\partial}{\partial z}h(w) \neq 0$  and for all  $z \neq w, z \in S^1, f(z) \neq f(w)$ . But, this implies that  $h(z) \neq f(w)$ , for all  $z \in \bar{D}$ . This implies that there is a nonempty open set  $U \subset D$  with the desired property.  $\square$

**Proposition 0.2.** *Suppose that  $h \in \mathbb{C}[z]$  satisfies the conclusion of Proposition 0.1, then any bounded operator  $S : L_a^2(D) \rightarrow L_a^2(D)$ , which commutes with  $T_h$  must be of the form  $T_f$ , for some  $f \in H^\infty(D)$ .*

The following proof is contained in [CF].

*Proof.* Recall that for any  $g(z) \in L_a^2(D)$ ,  $\langle g, K_z \rangle = g(z)$ , where  $K_z$  is the reproducing kernel. This gives  $T_h^* K_w = \overline{f(w)} K_w$ , in particular,  $T_{h(z)-h(w)}^* K_w = 0$ . Thus, since  $S^*$  and  $T_h^*$  commute, we have  $T_{f(z)-f(w)}^* (S^* K_w) = 0$ . Which means that  $S^* K_w$  is orthogonal to the image of  $T_{h(z)-h(w)}$ , which by our proposition is  $(z-w)L_a^2(D)$ , for all  $w \in U$ . But, since  $K_w$  is also orthogonal to the above, we have that  $S^* K_w = \psi(w) K_w$ , for all  $w \in U$ , where  $\psi(w)$  is some function on  $U$ . Thus,

$$\langle g, S^* K_w \rangle = \langle S(g), K_w \rangle = S(g)(w) = \overline{\psi(w)} g(w).$$

This implies that  $[S, T_z](g)|_U = 0$ , so  $S$  commutes with  $T_z$ , therefore  $S = T_\eta$ , for some bounded analytic  $\eta$ .  $\square$

*Proof of Theorem 0.1.* For any  $0 \leq i < m$ , consider bounded linear operators  $e_i, f_i : L_a^2(D) \rightarrow L_a^2(D)$  defined as follows:  $e_i(z^n) = z^{i+nm}$ ,  $f_i(z^{i+nm}) = z^n$  for all  $n$ . Thus,  $f_i$  is the composition of the orthogonal projection of  $L_a^2(D)$  on  $e_i(L_a^2(D))$  with  $e_i^{-1}$ . Let us put  $T_{i,j} = f_j S e_i$ . It is clear that  $S_{i,j}$  commutes with  $T_{h_1(z)}$ . So  $S_{i,j}$  is given by  $T_{\psi_{i,j}}$ , for some bounded analytic  $\psi_{i,j}$ , by propositions 0.1, 0.2, this implies that  $S$  commutes with  $T_{z^m}$ . If in addition,  $S$  is compact, then so are operators  $e_i S f_j = T_{\psi_{i,j}} : L_a^2(D) \rightarrow L_a^2(D)$ , which forces  $\psi_{i,j} = 0$ , so  $e_i S f_j = 0$  for all  $i, j$ , so  $S = 0$ .  $\square$

Now we turn to the proof of Corollary 0.1.

**Lemma 0.1.** [C] *If  $S$  belongs to the Toeplitz algebra, then  $[T_z, S]$  is a compact operator.*

We recall the proof for the convenience of the reader.

*Proof.* Since compact operators form a two sided ideal in the algebra of bounded operators, it is enough to check that  $[T_f, T_z]$  is compact for any  $f \in L^\infty(D)$ . However  $[T_f, T_z] = H_z^* H_f$ , where  $H_f(g) = fg - T_f(g)$ ,  $H_f : L_a^2(D) \rightarrow L^2(D)^\perp$  is the Hankel operator with symbol  $f$ . It is well-known that the Hankel operator  $H_z$  is compact, so we are done.  $\square$

Now we can easily proof Corollary 0.1. If  $S$  belongs to the Toeplitz algebra, and it commutes with  $T_h$  for a nonconstant  $h \in \mathbb{C}[z]$ , then by the above  $[T_z, S]$  is compact. But, since  $[T_h, [T_z, S]] = 0$ , Theorem 0.1 implies that  $T_z$  commutes with  $S$ , forcing  $S$  to be of the form  $T_\psi$ , for some bounded holomorphic  $\psi$ .

It is natural to ask if Theorem 0.1 and Corollary 0.1 hold for arbitrary nonconstant bounded holomorphic functions. A positive indication in this direction is provided by a well-known theorem of Axler-Cuckovic-Rao [ACR], which says that if  $T_g$  commutes with  $T_f$  for bounded  $g$  and nonconstant holomorphic  $f$  (in fact for an arbitrary bounded domain, not just the unit disk  $D$ ), then  $g$  must be holomorphic. However, both Theorem 0.1 and Corollary 0.1 fail for arbitrary nonconstant bounded holomorphic functions (contrary to what is claimed in [L]). We present two examples below.

The first example, due to Trieu Le, shows that Theorem 0.1 fails for arbitrary holomorphic functions. Indeed, let  $g : D \rightarrow D$  be a Mobius automorphism of the unit disc  $g(z) = \frac{z-a}{1-\bar{a}z}$ ,  $a \in D, a \neq 0$ , and let  $T : L_a^2 \rightarrow L_a^2$  be a bounded operator. Let us denote by  $gT : L_a^2(D) \rightarrow L_a^2(D)$  an operator defined as follows  $gT(f(z))(w) = T(f(g^{-1}(z)))(g(w))$ ,  $f \in L_a^2(D), w \in D$ . Let us take an operator  $T$ , which commutes with  $z^n$ , such that  $T$  is not a Toeplitz operator with an analytic symbol. Then,  $gT$  commutes with  $T_{g^n}$  and is not a Toeplitz operator with an analytic symbol. However,  $g^n$  satisfies the condition of the proposition, namely, it cannot be written as a holomorphic function of  $z^m$ , for any  $m > 1$ .

The next example shows that even Corollary 0.1 is false for arbitrary nonconstant holomorphic functions. Indeed, an example of Cowen [Co] provides a bounded holomorphic function whose Toeplitz symbol commutes with a compact operator on the Hardy space. But exactly the same example works for the Bergman space setting. Let us recall Cowen's example for the convenience of the reader. Let  $\sigma(z) = (i-1)(1+z)^{-\frac{1}{2}}$ . Then  $J(z) = \sigma^{-1}(\sigma(z) + 2\pi i)$  maps  $D$  to itself, continuously extends to the boundary of  $D$  and  $J(\bar{D}) \subset D \cup -1$ ,  $J(-1) = -1$ . Let  $f(z) = \exp(\sigma(z)) - \exp(\sigma(0))$ , then  $f$  is a bounded analytic function on  $D$  and  $f(J(z)) = f(z)$  for all  $z \in D$ . Denote by  $C_J$  the composition operator of  $J$ , so  $C_J(f) = f(J)f \in L_a^2(D)$ . Now claim is that the operator  $L = C_J T_{z+1}$  is compact. Indeed, let  $\|z^n g_n\| = 1$ . For any  $\epsilon > 0$ , let  $K_\epsilon \subset D$  denote the set of all  $z$ , such that  $|1 + J(z)| \geq \epsilon$ . Clearly  $\overline{J(K_\epsilon)}$  is compact, so there is  $0 < \delta < 1$ , such that  $J(z) < \delta < 1$  for

all  $z \in K_\epsilon$ . It is also clear that  $\|g_n\| \leq \sqrt{n+1}$ . We have

$$\int_{K_\epsilon} |j^n(z)g_n(J(z))(J(z)+1)|^2 d\mu \leq 2\delta^{2n}\|C_J\|\sqrt{n+1},$$

which clearly goes to 0 as  $n \rightarrow \infty$ . On the other hand,

$$\int_{D \setminus K_\epsilon} |j^n(z)g_n(J(z))(J(z)+1)|^2 d\mu \leq \epsilon\|C_J\|.$$

Therefore, we can conclude that  $\|L_{z^n L_a^2}\| \rightarrow 0$ , so  $L$  is compact and it commutes with  $T_f$ . But as it is well-known, all compact operators belong to the Toeplitz algebra, and since a nonzero compact operator can not be a Toeplitz operator with an analytic symbol, we see that Corollary 0.1 is false for arbitrary analytic functions.

Finally, we present a partial result for centralizers of  $T_f$ , where  $f$  is a nonconstant bounded holomorphic function. To state our result, we must recall that any function  $g \in L^2(D)$  admits a polar decomposition

$$g(re^{it}) = \sum_{k=-\infty}^{+\infty} e^{ikt} g_k(r),$$

where  $f_r$  are radial functions.

We have the following

**Proposition 0.3.** *Suppose that an operator  $S$  belongs to the algebra generated by Toeplitz operators of the form  $T_g, g_k = 0$  for  $k < 0$ . If  $S$  commutes with  $T_f$ , where  $f$  is bounded holomorphic function such that  $f'(0) \neq 0$ , then  $S = T_\psi$  for some  $\psi \in H^\infty(D)$ .*

*Proof.* Recall that by a computation from [CL],  $T_{e^{ikt}g(r)}(z^n)$  is a multiple of  $z^{n+k}$ , for all  $k, n$ . In particular, for any  $k \geq 0$   $\langle S(z^n), z^k \rangle = 0$  as long as  $n \gg 0$ . Let  $f = a_0 + a_1 z + z^2 f_1, a_1 \neq 0, m > 0, f_1 \in H^\infty(D)$ . It suffices to show that  $S$  commutes with  $z^m$ . Notice that for any  $z^k$ , there exists a polynomial  $\phi_k(z)$ , such that  $z - \phi_k(f(z)) \in z^k H^\infty(D)$ . This implies that for any  $m, l \geq 0$

$$\langle [S, T_z](z^l), z^n \rangle = \langle [S, T_z - T_{\phi_k(f(z))}](z^l), z^n \rangle = 0$$

for  $k \gg 0$ . So,  $T_z$  commutes with  $S$ , and we are done.  $\square$

**Acknowledgements.** I am enormously grateful to Trieu Le for many interesting discussions, in particular for telling me about the commuting problem for the Toeplitz algebra. I would like to thank Zeljko Cuckovic for showing me his paper [CF], ideas from which played the crucial role. Special thanks are due to Mitya Boyarchenko for his key insight with the proof of proposition 0.1. I also would like to thank Abdel Yousef for telling me about operators with quasi-homogeneous symbols.

## REFERENCES

- [ACR] S. Axler, Z. Cuckovic, N. Rao *Commutants of analytic Toeplitz operators on the Bergman space*, Proc. Amer. Math. Soc. 128 (2000), no. 7, 1951-1953.
- [Co] C. Cowen, *An analytic Toeplitz operator that commutes with a compact operator and a related class of Toeplitz operators* J. Funct. Anal. 36 (1980), no. 2, 169-184.
- [C] Z. Cuckovic, *Commutants of Toeplitz operators*, Pacific J. Math (1994) vol 162, No. 2. 277–285
- [CF] Z. Cuckovic, D. Fan *Commutants of Toeplitz operators on the ball and annulus* Glasgow Math. J. 37 (1995) no 3. 303–309.
- [CL] Z. Cuckovic, I. Louhichi *Finite rank commutators and semicommutators of quasi-homogeneous Toeplitz operators* Complex Anal. Oper. Theory 2 (2008), no. 3, 429–439.
- [L] Y. Li *The Commutant of Analytic Toeplitz Operators on Bergman Space* Acta Math. Sin. (Engl. Ser.) 24 (2008), no. 10, 1737-1750.
- [Q] J. R. Quine, *On the self-intersections of the image of the unit circle under a polynomial mapping* Proceeding of AMS (1973), Vol. 39 133–140

THE UNIVERSITY OF TOLEDO, DEPARTMENT OF MATHEMATICS, TOLEDO, OHIO, USA  
*E-mail address:* atikara@utnet.utoledo.edu